Adomian Decomposition and Variational Iteration Methods for Solving a Problem Arising in Modelling of Biological Species Living Together

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Adomian decomposition and He's variational iteration methods are analytical techniques, which can be used for solving various kinds of problems. The main property of these methods is in their flexibility and ability to solve nonlinear equations accurately. In this paper, the decomposition method and the variational iteration technique are explained, and their merits as well as their drawbacks are discussed. Then a new implementation of these methods is proposed, which yields an approximate solution with high accuracy in large regions and less computational efforts. A system of integro-differential equations arising in modelling of the biological species [1] living together is employed to show how these techniques work efficiently.

Key words: Initial Value Problems; Local Adomian Decomposition Method; Local He's Variational Iteration Method; Integro-Differential Equations; Closed Form Solution; Modelling of the Biological Species Living Together.

1. Introduction

Numerical methods can be efficient and accurate tools for solving various types of equations. These methods are based on the discretization techniques which are implemented with limiting considerations to guarantee the stability and convergence. This feature is the main problem in the numerical methods, especially for nonlinear equations, because the small perturbation in the given data which is not avoidable in the numerical schemes causes large amount of perturbations in the approximate solution. Although analytical methods are not always applicable, these schemes are efficient for avoiding the perturbation.

In this work, we consider two efficient methods for finding an approximate solution of a large class of linear and nonlinear problems. These techniques are known as the Adomian decomposition method (ADM) and the variational iteration method (VIM).

The ADM has been proposed by the American mathematician G. Adomian. It is based on the search for a solution in the form of a series and on decomposing the nonlinear operator in to a series in which the terms are calculated recursively using Adomian polynomials [2]. It can be used for many kinds of differential equations [3]. For solving problems in calculus of variations, this scheme is investigated in [4]. Furthermore, this method is used for finding the solution of higher-order differential equations [5 – 7]. The Adomian decomposition method is employed in [8] to solve the pantograph equation which arises in the modelling of electrodynamic problems.

This approach is proposed in [9] to solve the Eikonal nonlinear partial differential equation. This approach which accurately computes the series solution is of great interest to engineering, physics, biology, and so on. This scheme provides the solution in a rapidly convergent series with components that can be elegantly computed [10 – 12]. In [13] the Adomian decomposition method and the modified Adomian decomposition method are improved for solving approximately homogeneous and inhomogeneous two-dimensional heat equations using Padé approximations. Also this approach was used to find approximate solutions of the coupled Burgers equations [13]. The results show that these techniques increase efficiently the accuracy of approximate solutions and lead to convergence with a rate faster than using the Adomian decomposition and the modified Adomian decomposition methods [13]. Also in [14] authors used the Adomian-

Padé and modified Adomian-Padé techniques for solving approximately inhomogeneous systems of Volterra functional equations. The results demonstrate that the ADM-Padé (modified ADM-Padé) technique gives the approximate solution with faster convergence rate and higher accuracy than using the standard ADM (modified ADM) [14]. The well-known delay differential equation, namely pantograph equation of order m, is investigated in [15] using the Adomian decomposition method.

Using the ADM, we get a series solution. The series often coincides with Taylor expansion of the true solution. Although this series has high accuracy in a very small region, it does not have desired accuracy in the wider region.

To remedy this drawback, we introduce the local Adomian decomposition method (LADM). This new idea is based on using the Adomian decomposition method repeatedly in the prescribed subregions.

The well known He's variational iteration method is a very powerful method for solving a large amount of problems. This technique is developed by the Chinese researcher J. H. He [16]. This scheme is applied to find the solution of several classes of variational problems [17]. Application of this method to Helmholtz equation is investigated in [18]. The variational iteration method is used for solving autonomous ordinary differential systems in [16]. In [19], this method has been used for studying nonlinear oscillators. The efficiency of this method for solving various type of problems is shown in [1, 18, 20 – 25]. The variational iteration procedure is employed in [26] to solve the nonlinear mixed Volterra-Fredholm integral equations. This method is applied in [27] to find the solution of telegraph and fractional telegraph equations. Also the use of this method for solving linear fractional partial differential equations arising from fluid mechanics is presented in [28, 29]. The study of singular value problems of Lane-Emden type has been carried out in [30]. For more information in this field see [31, 32].

The variational iteration method is a very efficient method for solving initial value problems. It provides a sequence which converges to the solution of the problems without discretization of the variables. Although the method is rapidly convergent, it is only accurate in a small region. To find the solution in the larger regions, more terms of the sequence should be found which is impossible or difficult.

In this study we introduce a local variant of He's variational iteration method (LVIM), which is proposed in [33]. The main idea behind the new method is the successive use of the variational iteration method in the subregions. This approach remedies the growth of the approximate solution in He's variational iteration method.

Integro-differential equations play an important role in modelling of physical and biological phenomena. In this paper the following nonlinear system of integro-differential equations which arises in modelling of the biological species living together [1] is considered:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p(t)\left[k_1 - \gamma_1 q(t) - \int_{t-T_0}^{t} f_1(t-\tau)q(\tau)\mathrm{d}\tau\right] + g_1(t), \quad k_1,\gamma_1 > 0,\ 0 \le t \le l, \tag{1}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t} = q(t)\left[-k_2 + \gamma_2 p(t) + \int_{t-T_0}^{t} f_2(t-\tau)p(\tau)\mathrm{d}\tau\right] + g_2(t), \quad k_2,\gamma_2 > 0,\ 0 \le t \le l, \tag{2}$$

with the initial conditions

$$p(0) = \alpha_1, \quad q(0) = \alpha_2,$$

where p and q are unknown, and f_1, f_2, g_1, and g_2 are given functions.

The rest part of this paper is organized as follows: In Section 2, the decomposition method is reviewed and the new technique is introduced. Descriptions of He's variational iteration method and localization technique are carried out in Section 3. In Section 4, implementation of these methods on the system of integro-differential equations is given. Some examples of integro-differential equations [1] have been presented to show the efficiency of the new methods in Section 5. Finally, a conclusion is drawn in Section 6.

2. Adomian Method and its New Implementation Based on the Localization Technique

Let us consider a problem in an operator form [2]:

$$Ly - Ny = f(t), \quad t \ge t_0, \tag{3}$$

where generally L is an invertible linear operator, which is taken as highest-order derivative, N represents the nonlinear operator which contains less-order derivatives, and f is a given function. Operating L^{-1} on two sides of (3) yields

$$L^{-1}Ly = L^{-1}Ny + L^{-1}f.$$

An equivalent expression is

$$y = \Phi + L^{-1}Ny + L^{-1}f,$$

where Φ is the integration constant. If L is a second-order derivative, then L^{-1} is a two-fold integration and is defined for a function $g(t)$ as follows:

$$L^{-1}(g(t)) = \int_{t_0}^{t}\int_{t_0}^{x_2} g(x_1)\mathrm{d}x_1\mathrm{d}x_2.$$

Applying the operator $L^{-1} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}$ to both sides of (3) yields

$$y(t) - y(t_0) - y'(t_0)t + y'(t_0)t_0 = L^{-1}N(y) + L^{-1}f,$$

or equivalently

$$y = \Phi + L^{-1}Ny + L^{-1}f, \tag{4}$$

where $\Phi = y(t_0) + y'(t_0)t - y'(t_0)t_0$. According to the decomposition procedure of Adomian, the solution y is represented as the infinite sum of series

$$y = \sum_{n=0}^{\infty} y_n(t) \tag{5}$$

and takes the nonlinear expression $N(y)$ by the infinite series of Adomian polynomial, given by

$$N(y) = \sum_{n=0}^{\infty} N_n. \tag{6}$$

The components N_n are appropriate Adomian polynomials, which are calculated in the following form [3]:

$$N_n(y_0, y_1, \ldots, y_n) = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} N\left(\sum_{k=0}^{\infty}\lambda^k y_k\right)\Bigg|_{\lambda=0}, \quad n \geq 0.$$

Putting (6) and (5) into (4) gives

$$\sum_{n=0}^{\infty} y_n = \Phi + L^{-1}f + L^{-1}\sum_{n=0}^{\infty} N_n.$$

Each term of series (5) is given by the recurrent relation

$$y_0 = \Phi + L^{-1}f, \quad y_n = L^{-1}N_{n-1}, \quad n \geq 1. \tag{7}$$

However, in practice all terms of the series (5) can not be determined, and the solution will be approximated by a truncated series

$$\Psi_n = \sum_{i=0}^{n} y_i(t), \quad n \geq 0. \tag{8}$$

Since this method finds a solution of the given equation without any requirement to discretization and linearization, we should use symbolic computation for finding approximate solution.

Although the series solutions converge rapidly only in a small region, in a wider region, they may have very slow convergence, and then their truncations yield inaccurate results. For description of the behaviour of Ψ_n in a large region, we generally consider the initial value problem

$$y' = N(y) + f(t) \tag{9}$$

with the initial term $y_0 = y(t_0)$. In fact, problem (9) is the general form of the integro-differential equations (1) and (2). In this case $L = \frac{\mathrm{d}}{\mathrm{d}t}$ and implementing (7) yields

$$\begin{aligned} y_0(t) &= y_0 + \int_{t_0}^{t} f(s)\mathrm{d}s, \\ y_{n+1} &= \int_{t_0}^{t} N_n(y_0, \ldots, y_n)\mathrm{d}s, \quad n \geq 0. \end{aligned} \tag{10}$$

Thus, according to (10) for $k \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{i=1}^{k} y_i &= \sum_{i=1}^{k}\int_{t_0}^{t} N_{i-1}(y_0, \ldots, y_{i-1})\mathrm{d}s \\ &= \int_{t_0}^{t}\sum_{i=1}^{k} N_{i-1}(y_0, \ldots, y_{i-1})\mathrm{d}s. \end{aligned} \tag{11}$$

Therefore (10) and (11) yield

$$\sum_{i=0}^{k} y_i = y_0 + \int_{t_0}^{t} f(s)\mathrm{d}s + \int_{t_0}^{t}\sum_{i=1}^{k} N_{i-1}(y_0, \ldots, y_{i-1})\mathrm{d}s. \tag{12}$$

Also the differential equation (9) can be written as

$$y = y_0 + \int_{t_0}^{t}(N(y(s)) + f(s))\mathrm{d}s. \tag{13}$$

Finally, using (12) and (13), we deduce

$$\begin{aligned} &\left|y - \sum_{i=0}^{k} y_i\right| = \\ &\left|\int_{t_0}^{t}\left(N(y(s)) - \sum_{i=1}^{k} N_{i-1}(y_0, \ldots, y_{i-1})\right)\mathrm{d}s\right| \\ &\leq \int_{t_0}^{t}\left|N(y(s)) - \sum_{i=1}^{k} N_{i-1}(y_0, \ldots, y_{i-1})\right|\mathrm{d}s \end{aligned}$$

$$\leq (t-t_0)\left(\left\| N(y) - \sum_{i=1}^{k} N_{i-1}(y_0,\ldots,y_{i-1})\right\|_\infty\right)$$
$$\leq (t-t_0)\|N(y)\|_\infty + (t-t_0)\sum_{i=1}^{k}\|N_{i-1}(y_0,\ldots,y_{i-1})\|_\infty.$$

According to (9), we get $N(y) = y' - f(t)$. Hence,

$$\left\| y - \sum_{i=0}^{k} y_i \right\|_\infty \leq (t-t_0)\Bigg(\|y'\|_\infty + \|f\|_\infty + \sum_{i=1}^{k}\|N_{i-1}(y_0,\ldots,y_{i-1})\|_\infty\Bigg). \quad (14)$$

On the other hand (10) yields

$$y'_{n+1} = N_n(y_0,\ldots,y_n), \quad n \geq 0.$$

So, (14) gives

$$\left\| y - \sum_{i=0}^{k} y_i \right\|_\infty \leq (t-t_0)\left(\|y'\|_\infty + \|f\|_\infty + \sum_{i=1}^{k}\|y'_i\|_\infty\right), \quad (15)$$

which results that the maximum error increases by the growth of t. Moreover, the maximum error depends on f and y'_i for $1 \leq i \leq k$ and y'. Therefore, if f and y'_i converge to infinity by the growth of t, the maximum error converges rapidly to infinity.

In the new method we try to reduce the error of this method by controlling the right hand side in (15). According to (15) an idea for reducing the error is to decrease k and $t - t_0$. For a moderate n_0, $\sum_{i=0}^{n_0} y_i^0$ is found as an approximation of the solution in the region $[t_0, t_1]$ by means of

$$y_0^0(t) = y(t_0) + \int_{t_0}^{t} f(s)\mathrm{d}s, \quad y_{i+1}^0 = \int_{t_0}^{t} N_i^0 \mathrm{d}s,$$
$$0 \leq i \leq n_0 - 1.$$

Therefore, we write

$$y^0 = \sum_{i=0}^{n_0} y_i^0.$$

Then the initial value problem (9) is considered for $t > t_1$ with the initial condition $y^0(t_1)$. So the new solution is written as

$$y_0^1(t) = y^0(t_1) + \int_{t_1}^{t} f(s)\mathrm{d}s, \quad y_{i+1}^1 = \int_{t_1}^{t} N_i^1 \mathrm{d}s,$$
$$0 \leq i \leq n_1 - 1,$$

with the initial term $y_0^1 = y^0(t_1)$. Hence,

$$y^1 = \sum_{i=0}^{n_1} y_i^1. \quad (16)$$

This is an approximate solution in $[t_1, t_2]$ where t_2 is chosen in a region so that $\sum_{i=0}^{n_1} y_i^1$ be sufficiently accurate in $[t_1, t_2]$.

Generally for $t \in [t_m, t_{m+1}]$, we have $y_0^m = y^{m-1}(t_m)$ and

$$y_0^m(t) = y_0^m + \int_{t_m}^{t} f(s)\mathrm{d}s, \quad y_{i+1}^m = \int_{t_m}^{t} N_i^m \mathrm{d}s,$$
$$0 \leq i \leq n_m - 1.$$

Therefore,

$$y^m = \sum_{i=0}^{n_m} y_i^m,$$

where the choice of n_i depends on $h_i = t_{i+1} - t_i$. This procedure is called local Adomian decomposition method (LADM). For convenience, we can assume $n_i = n$ and $h_i = h$ for $i = 0, \ldots, m$, then we set

$$F_0 = a + \int_b^t f(s)\mathrm{d}s, \quad F_{i+1} = \int_b^t N_i(F_0,\ldots,F_i)\mathrm{d}s,$$
$$0 \leq i \leq n-1.$$

So $F_n = F_n(a,b)$. Therefore, the solution of the problem is provided in the piecewise form as

$$y_n(t) = F_n(y_n(t_i), t_i) \ \text{ if } \ t_i \leq t \leq t_{i+1},$$
$$\text{for } \ i = 0, 1, 2, \ldots,$$

where $t_i = t_0 + ih$ for $i = 1, 2, \ldots$. Since it needs only a symbolic computation in finding F_n for a small n, the new technique provides the solution of the problem in a short time. We can apply the above scheme on the system of equations

$$L(y_l) - N_l(y_1,\ldots,y_s) = f_l, \quad l = 1, 2, \ldots, s, \quad (17)$$

where N_l for $l = 1, \ldots, s$, are nonlinear operators and f_l for $l = 1, \ldots, s$, are given functions. Thus we have

$$y_l(t) = y_l(t_0) + L^{-1}f_l + L^{-1}N_l(y_1, y_2, \ldots, y_s),$$
$$l = 1, 2, \ldots, s.$$

By ADM, the following recursive relation is deduced:

$$y_{l0}(t) = y_l(t_0) + L^{-1}f_l(t), \quad y_{l(n+1)}(t) = L^{-1}N_{ln},$$
$$n \geq 0, \quad l = 1, 2, \ldots, s,$$

and the series solution is given in the following form:

$$y_l(t) = \sum_{n=0}^{\infty} y_{ln}(t), \quad l = 1,2,\dots,s.$$

Now we implement LADM. So for $t \in [t_m, t_{m+1}]$, we have $y_{l0}^m = y_l^{m-1}(t_m)$, for $l = 1,2,\dots,s$, and

$$y_{l0}^m(t) = y_{l0}^m + \int_{t_m}^t f_l(s)\mathrm{d}s, \quad y_{l(i+1)}^m = \int_{t_m}^t N_{li}^m \mathrm{d}s,$$
$$0 \le i \le n_m - 1.$$

Therefore, the approximate solution for $t \in [t_m, t_{m+1}]$ is

$$y_l^m(t) = \sum_{i=0}^{n_m} y_{li}^m(t). \quad l = 1,2,\dots,s.$$

3. Variational Iteration Method and the New Implementation

In this method the problem is considered as

$$Ly + Ny = g(t), \tag{18}$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is a given function. In this scheme, a correction functional is constructed by a general Lagrange multiplier, which can be identified optimally via the variational theory [20]. Using the variational iteration method (VIM), the following correction functional is considered:

$$y_{n+1} = y_n + \int_{t_0}^t \lambda(Ly_n(s) + N\tilde{y}_n(s) - g(s))\mathrm{d}s, \tag{19}$$

where the initial condition has been given in t_0, λ is Lagrange multiplier, the subscript n denotes the n-th approximation, $\tilde{y}_n$ is considered as a restricted variation which means that $\delta\tilde{y}_n = 0$ [34]. By taking the variation from both sides of the correction functional with respect to the independent variable y_n and imposing $\delta y_{n+1} = 0$, the stationary conditions are obtained. Using the stationary conditions, the optimal value of λ is identified and is classified for different forms of differential equations in [31].

Since this method avoids the discretization of the problem, it is possible to find a closed form solution without any round off error.

A description of VIM for solving integro-differential equations (1)–(2) is given in [1].

In this study we consider integro-differential equation (1) and (2) in the general form

$$y' = N(y) + g(t) \tag{20}$$

with the given initial condition $y(t_0)$. So for this problem He's variational iteration method is given in the form

$$y_{n+1} = y_n + \int_{t_0}^t \lambda(y_n'(s) - Ny_n(s) - g(s))\mathrm{d}s$$

with the initial term $y_0 = y(t_0)$.

In some problems finding more terms of the sequence (y_n) is difficult or impossible. So an accurate approximation is provided near the initial condition only. Namely, the approximate solution is accurate in $t_0 \le t \le t_1$. Since y_n has been found by successive integrations, it contains terms with high powers of t. Therefore, y_n converges to infinity faster than y as t increases. For studying the behaviour of y_n, VIM yields

$$y_{n+1} = y_n + \int_{t_0}^t \lambda(y_n'(s) - Ny_n(s) - g(s))\mathrm{d}s.$$

Also the differential equation can be written as

$$y = y(t_0) + \int_{t_0}^t (Ny(s) + g(s))\mathrm{d}s.$$

Therefore, we have

$$\begin{aligned}|y - y_{n+1}| = \Big| y(t_0) - y_n + \int_{t_0}^t \Big(&Ny(s) + g(s) \\ &+ \lambda Ny_n(s) + \lambda g(s) - \lambda y_n'(s)\Big)\mathrm{d}s\Big|.\end{aligned} \tag{21}$$

The integro-differential equations (1) and (2) can be written as $y' + \alpha y + f(y,t) = 0$, so the optimal value of λ according to [31] is $-\mathrm{e}^{\alpha(s-t)}$ for $t_0 \le s \le t$.

Hence, $-1 \le \lambda \le 0$ and $Ny = y' - g(t)$, and we can write

$$\begin{aligned}|y - y_{n+1}| &\le |y(t_0) - y_n| \\ &\quad + \int_{t_0}^t |y' + \lambda Ny_n(s) + \lambda g(s) - \lambda y_n'(s)|\mathrm{d}s \\ &\le |y(t_0) - y_n| + \int_{t_0}^t \big(|y'(s)| + |\lambda Ny_n(s)| \\ &\quad + |\lambda g(s)| + |\lambda y_n'(s)|\big)\mathrm{d}s.\end{aligned}$$

So we have

$$\begin{aligned}\|y - y_{n+1}\|_\infty \le \|y(t_0) - y_n\|_\infty + (t - t_0)\big(&\|y'\|_\infty \\ &+ \|Ny_n(s)\|_\infty + \|g\|_\infty + \|y_n'\|_\infty\big),\end{aligned} \tag{22}$$

which results that the maximum error increases by the growth of t, y'_n, y', and g. In fact, the approximate solution is not accurate in the larger regions.

In the new method using the approximations which have been resulted from less iterations, we try to overcome this problem. For an appropriate n_0, $y^0_{n_0}$ is found as an approximation of the solution of the problem in a small region $[t_0,t_1]$ by the sequence

$$y^0_{n+1} = y^0_n + \int_{t_0}^{t} \lambda (y'^0_n(s) - Ny^0_n(s) - g(s))\mathrm{d}s$$

with initial term $y^0_0 = y(t_0)$. Then the initial value problem (20) is considered for $t > t_1$ with the initial condition $y^0_{n_0}(t_1)$. The new sequence is written as

$$y^1_{n+1} = y^1_n + \int_{t_1}^{t} \lambda (y'^1_n(s) - Ny^1_n(s) - g(s))\mathrm{d}s$$

with the initial term $y^1_0 = y^0_{n_0}(t_1)$. Similarly for an integer n_1, y_{n_1} is an approximation of the solution in $[t_1,t_2]$ where t_2 is chosen in a way that $y^1_{n_1}$ be sufficient accurate in $[t_1,t_2]$.

This technique can be continued to find an accurate approximation $y^m_{n_m}$ in $[t_m,t_{m+1}]$, using the following sequence

$$y^m_{n+1} = y^m_n + \int_{t_m}^{t} \lambda (y'^m_n(s) - Ny^m_n(s) - g(s))\mathrm{d}s$$

with the initial term $y^m_0 = y^{m-1}_{n_{m-1}}(t_m)$. The values of n_i depend directly on $h_i = t_{i+1} - t_i$. We name the new method as local He's variational iteration method (LVIM).

Generally, the values of n_i, $i = 0,1,2,\ldots$ are not equal. For equal choice of n_i and h_i, a simple symbolic computation finds the solution easily. In fact, if $n_i = n$ and $h_i = h$ for $i = 0,1,2,\ldots$, we set

$$F_0 = a,\ \ F_{k+1} = F_k - \int_{b}^{t} \lambda_1 (F'_k(s) - NF_k(s) - g(s))\mathrm{d}s,$$
$$k = 0,\ldots n-1.$$

So $F_n = F_n(a,b)$. Therefore, the solution of the problem is provided in the piecewise form as

$$y_n(t) = F_n(y_n(t_i),t_i) \ \text{ if } \ t_i \le t \le t_{i+1},$$
$$\text{for } \ i = 0,1,2,\ldots,$$

where $t_i = t_0 + ih$ for $i = 1,2,\ldots$.

4. Solving a System of Integro-Differential Equations

In this section we first implement the scheme of Section 2 for solving (1) and (2), hence we write [1]

$$L^{-1} = \frac{\mathrm{d}}{\mathrm{d}t}, \quad y = (p,q)^t, \quad N(y) = (N_1(y), N_2(y))^t,$$
$$N_1((p,q)) = p(t)\left[k_1 - \gamma_1 q(t) - \int_{t-T_0}^{t} f_1(t-\tau)q(\tau)\mathrm{d}\tau\right],$$
$$N_2((p,q)) = q(t)\left[-k_2 + \gamma_2 p(t) + \int_{t-T_0}^{t} f_2(t-\tau)p(\tau)\mathrm{d}\tau\right].$$

Therefore, we find the solution in the following form:

$$p_0(t) = p(t_0) + \int_{t_0}^{t} g_1(s)\mathrm{d}s,$$
$$q_0(t) = q(t_0) + \int_{t_0}^{t} g_2(s)\mathrm{d}s,$$
$$p_{n+1}(t) = L^{-1}N_{1n}, \quad q_{n+1}(t) = L^{-1}N_{2n}, \quad n \ge 0.$$

Thus, we obtain n-th approximations of p and q in form

$$P_n = \sum_{i=0}^{n} p_i, \quad Q_n = \sum_{i=0}^{n} q_i.$$

A description of Adomian's procedure for solving (1) and (2), is given in [35]. Using LADM, n-th approximations of p and q are

$$\bar{P}_n(t) = P^m_n(t) \ \text{ if } \ t_m \le t \le t_{m+1},$$
$$\text{for } \ m = 0,1,2,\ldots,$$
$$\bar{Q}_n(t) = Q^m_n(t) \ \text{ if } \ t_m \le t \le t_{m+1},$$
$$\text{for } \ m = 0,1,2,\ldots,$$

where P^m_n and Q^m_n are approximate solutions in $t_m \le t \le t_{m+1}$ which are found using ADM and $t_i = t_0 + ih$ for $i = 1,2,\ldots$. According to VIM, the sequences p_n and q_n are constructed such that these sequences converge to the exact solution, which are calculated by a correction functional as follows:

$$p_{n+1}(t) = p_n(t) + \int_0^t \lambda_1 \left[\frac{\mathrm{d}p_n}{\mathrm{d}s} - k_1 p_n(s) + \gamma_1 \tilde{p}_n(s) q_n(s) + \tilde{p}_n(s)\int_{s-T_0}^{s} f_1(s-\tau)q_n(\tau)\mathrm{d}\tau - g_1(s)\right]\mathrm{d}s, \quad (23)$$

$$q_{n+1}(t) = q_n(t) + \int_0^t \lambda_2 \left[\frac{\mathrm{d}q_n}{\mathrm{d}s} + k_2 q_n(s) - \gamma_2 \tilde{q}_n(s) p_n(s) - \tilde{q}_n(s)\int_{s-T_0}^{s} f_2(s-\tau)p_n(\tau)\mathrm{d}\tau - g_2(s)\right]\mathrm{d}s. \quad (24)$$

This system has been solved by using VIM [1]. Taking variations of (23) and (24) with respect to p_n and q_n, respectively, and making the correction functional stationary, we obtain

$$\delta p_{n+1} = 0, \quad \delta q_{n+1} = 0.$$

Therefore, the Lagrange multipliers λ_1 and λ_2 can be identified as [31]:

$$\lambda_1(s) = -\mathrm{e}^{-k_1(s-t)}, \quad \lambda_2(s) = -\mathrm{e}^{k_2(s-t)}.$$

Then we obtain the following variational iteration formulas:

$$p_{n+1}(t) = p_n(t) - \int_0^t \mathrm{e}^{-k_1(s-t)} \left[\frac{\mathrm{d}p_n}{\mathrm{d}s} - k_1 p_n(s) + \gamma_1 \tilde{p}_n(s) q_n(s) + \tilde{p}_n(s) \int_{s-T_0}^{s} f_1(s-\tau) q_n(\tau) \mathrm{d}\tau - g_1(s) \right] \mathrm{d}s, \quad (25)$$

$$q_{n+1}(t) = q_n(t) - \int_0^t \mathrm{e}^{k_2(s-t)} \left[\frac{\mathrm{d}q_n}{\mathrm{d}s} + k_2 q_n(s) - \gamma_2 \tilde{q}_n(s) p_n(s) - \tilde{q}_n(s) \int_{s-T_0}^{s} f_2(s-\tau) p_n(\tau) \mathrm{d}\tau - g_2(s) \right] \mathrm{d}s. \quad (26)$$

So n-th approximations of p and q using LVIM are

$$\bar{p}_n(t) = p_n^m(t) \;\text{ if }\; t_m \le t \le t_{m+1}, \;\text{ for }\; m = 0,1,2,\ldots,$$
$$\bar{q}_n(t) = q_n^m(t) \;\text{ if }\; t_m \le t \le t_{m+1}, \;\text{ for }\; m = 0,1,2,\ldots,$$

where $t_i = t_0 + ih$ for $i = 1,2,\ldots$. According to the description of Sections 2 and 3, we can improve the accuracy of these methods via LADM and LVIM techniques.

5. Numerical Examples

As an application of the new methods, we implement them on some integro-differential equations as the test problems [1].

5.1. Example 1.

In this example we consider the system (1) and (2) with

$$f_1(t) = 1, \quad f_2(t) = t - 1,$$
$$k_1 = 1, \quad k_2 = 2, \quad \gamma_1 = \frac{1}{3}, \quad \gamma_2 = 1,$$
$$T_0 = \frac{1}{2}, \quad \alpha_1 = 1, \quad \alpha_2 = 0,$$

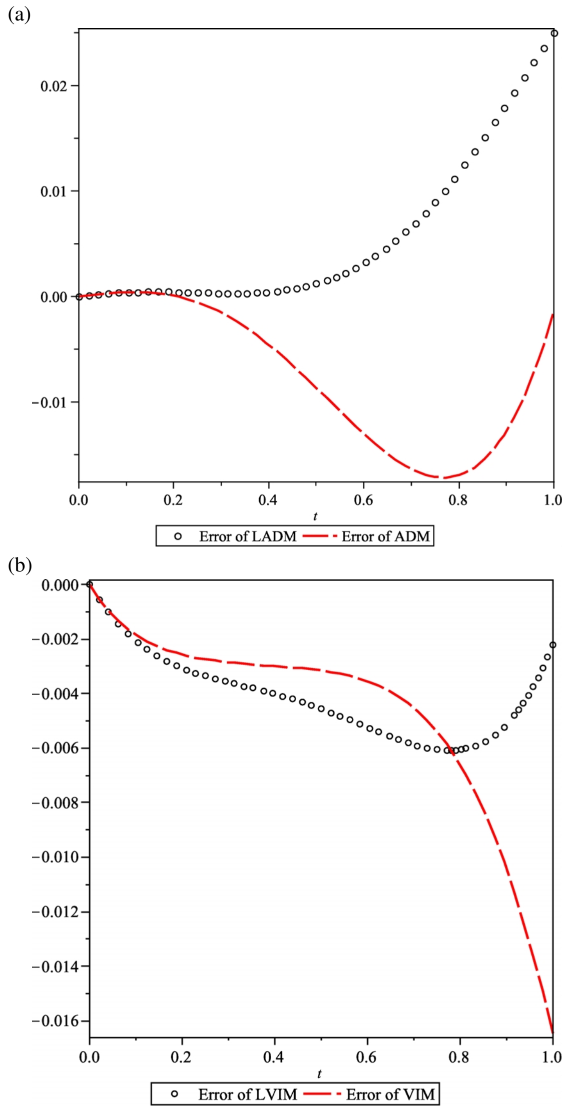

Fig. 1 (colour online). Plot of $P_2(t) - (-3t+1)$ and $\bar{P}_2(t) - (-3t+1)$ (a) for the Adomian method and LADM with $h = 0.001$ also, $p_2(t) - (-3t+1)$ and $\bar{p}_2(t) - (-3t+1)$ (b) for VIM and the new technique (LVIM) with $h = 0.001$.

$$g_1(t) = -\frac{5}{2}t^3 + \frac{49}{12}t^2 + \frac{17}{12}t - \frac{23}{6},$$

and

$$g_2(t) = \frac{15}{8}t^3 - \frac{1}{4}t^2 + \frac{3}{8}t - 1.$$

The exact solution is

$$p(t) = -3t + 1, \quad q(t) = t^2 - t.$$

Using ADM, we found two terms of iteration. Finding more terms causes consuming high computer memory.

Table 1. Results from ADM, LADM, VIM, LVIM in Example 1 for error of p.

t	$\lvert P_2(t)-p(t)\rvert$	$\lvert \bar{P}_2(t)-p(t)\rvert$	$\lvert p_2(t)-p(t)\rvert$	$\lvert \bar{p}_2(t)-p(t)\rvert$
0.1	0.4406×10^{-3}	0.4099×10^{-3}	0.1852×10^{-2}	0.2065×10^{-2}
0.2	0.8636×10^{-4}	0.4422×10^{-3}	0.2573×10^{-2}	0.3073×10^{-2}
0.3	0.1527×10^{-2}	0.3431×10^{-3}	0.2860×10^{-2}	0.3590×10^{-2}
0.4	0.4547×10^{-2}	0.4635×10^{-3}	0.3003×10^{-2}	0.4015×10^{-2}
0.5	0.8650×10^{-2}	0.1246×10^{-2}	0.3162×10^{-2}	0.4561×10^{-2}
0.6	0.1300×10^{-1}	0.3166×10^{-2}	0.3551×10^{-2}	0.5247×10^{-2}
0.7	0.1633×10^{-1}	0.6615×10^{-2}	0.4528×10^{-2}	0.5885×10^{-2}
0.8	0.1694×10^{-1}	0.1173×10^{-1}	0.6615×10^{-2}	0.6067×10^{-2}
0.9	0.1276×10^{-1}	0.1821×10^{-1}	0.1041×10^{-1}	0.5143×10^{-2}
1.0	0.1290×10^{-2}	0.2498×10^{-1}	0.1644×10^{-1}	0.2198×10^{-2}

Table 2. Results from ADM, LADM, VIM, LVIM in Example 1 for error of q.

t	$\lvert Q_2(t)-q(t)\rvert$	$\lvert \bar{Q}_2(t)-q(t)\rvert$	$\lvert q_2(t)-q(t)\rvert$	$\lvert \bar{q}_2(t)-q(t)\rvert$
0.1	0.4406×10^{-3}	0.4099×10^{-3}	0.1465×10^{-3}	0.1425×10^{-3}
0.2	0.8636×10^{-4}	0.4422×10^{-3}	0.4027×10^{-3}	0.3414×10^{-3}
0.3	0.1527×10^{-2}	0.4635×10^{-3}	0.6223×10^{-3}	0.4077×10^{-3}
0.4	0.4547×10^{-2}	0.4635×10^{-3}	0.6904×10^{-3}	0.3045×10^{-3}
0.5	0.8650×10^{-2}	0.1246×10^{-2}	0.4371×10^{-3}	0.8561×10^{-4}
0.6	0.1300×10^{-1}	0.3166×10^{-2}	0.3700×10^{-3}	0.1561×10^{-3}
0.7	0.1633×10^{-1}	0.6615×10^{-2}	0.1984×10^{-2}	0.3294×10^{-3}
0.8	0.1694×10^{-1}	0.1173×10^{-1}	0.4600×10^{-2}	0.3772×10^{-3}
0.9	0.1276×10^{-1}	0.1821×10^{-1}	0.8237×10^{-2}	0.3032×10^{-3}
1.0	0.1290×10^{-2}	0.2498×10^{-1}	0.1256×10^{-1}	0.1960×10^{-3}

Table 3. Results from ADM, LADM, VIM, LVIM in Example 2 for error of p.

t	$\lvert P_2(t)-p(t)\rvert$	$\lvert \bar{P}_2(t)-p(t)\rvert$	$\lvert p_2(t)-p(t)\rvert$	$\lvert \bar{p}_2(t)-p(t)\rvert$
0.1	0.5894×10^{-6}	0.1627×10^{-6}	0.1482×10^{-7}	0.5327×10^{-7}
0.2	0.3666×10^{-4}	0.4998×10^{-5}	0.9407×10^{-6}	0.7147×10^{-6}
0.3	0.2984×10^{-3}	0.3689×10^{-4}	0.6566×10^{-5}	0.6114×10^{-5}
0.4	0.1283×10^{-2}	0.1336×10^{-3}	0.2894×10^{-4}	0.2838×10^{-4}
0.5	0.3956×10^{-2}	0.3514×10^{-3}	0.9841×10^{-4}	0.9042×10^{-4}
0.6	0.9914×10^{-2}	0.7690×10^{-3}	0.2759×10^{-3}	0.2274×10^{-3}
0.7	0.2155×10^{-1}	0.1494×10^{-2}	0.6664×10^{-3}	0.4875×10^{-3}
0.8	0.4227×10^{-1}	0.2675×10^{-2}	0.1431×10^{-2}	0.9323×10^{-3}
0.9	0.7659×10^{-1}	0.4514×10^{-2}	0.2801×10^{-2}	0.1637×10^{-2}
1.0	0.1304	0.7285×10^{-1}	0.5092×10^{-2}	0.2694×10^{-2}

Table 4. Results from ADM, LADM, VIM, LVIM in Example 2 for error of q.

t	$\lvert Q_2(t)-q(t)\rvert$	$\lvert \bar{Q}_2(t)-q(t)\rvert$	$\lvert q_2(t)-q(t)\rvert$	$\lvert \bar{q}_2(t)-q(t)\rvert$
0.1	0.1627×10^{-4}	0.1329×10^{-6}	0.1847×10^{-8}	0.2506×10^{-7}
0.2	0.2502×10^{-3}	0.5812×10^{-6}	0.8553×10^{-6}	0.2733×10^{-7}
0.3	0.1233×10^{-2}	0.2480×10^{-5}	0.7296×10^{-5}	0.6491×10^{-8}
0.4	0.3818×10^{-2}	0.7962×10^{-5}	0.3474×10^{-4}	0.9162×10^{-7}
0.5	0.9163×10^{-2}	0.1990×10^{-5}	0.1205×10^{-3}	0.6056×10^{-6}
0.6	0.1871×10^{-1}	0.4182×10^{-4}	0.3386×10^{-3}	0.2355×10^{-5}
0.7	0.3419×10^{-1}	0.7785×10^{-4}	0.8147×10^{-3}	0.6764×10^{-5}
0.8	0.5749×10^{-1}	0.1327×10^{-3}	0.1740×10^{-2}	0.1587×10^{-4}
0.9	0.9065×10^{-1}	0.2117×10^{-3}	0.3382×10^{-2}	0.3224×10^{-4}
1.0	0.1356	0.3209×10^{-3}	0.6089×10^{-2}	0.5871×10^{-4}

Table 5. Results from ADM, LADM, VIM, LVIM in Example 3 for error of p.

t	$\lvert P_2(t)-p(t)\rvert$	$\lvert \bar{P}_2(t)-p(t)\rvert$	$\lvert p_2(t)-p(t)\rvert$	$\lvert \bar{p}_2(t)-p(t)\rvert$
0.1	0.6353×10^{-5}	0.2190×10^{-5}	0.1929×10^{-4}	0.1036×10^{-4}
0.2	0.6441×10^{-4}	0.7332×10^{-5}	0.4860×10^{-4}	0.2670×10^{-4}
0.3	0.2708×10^{-3}	0.1501×10^{-4}	0.8455×10^{-4}	0.3855×10^{-4}
0.4	0.7558×10^{-3}	0.2457×10^{-4}	0.5058×10^{-3}	0.4436×10^{-4}
0.5	0.1654×10^{-2}	0.3481×10^{-4}	0.1218×10^{-2}	0.4592×10^{-4}
0.6	0.3078×10^{-2}	0.4467×10^{-4}	0.2130×10^{-2}	0.4569×10^{-4}
0.7	0.5100×10^{-2}	0.5362×10^{-4}	0.3098×10^{-2}	0.4560×10^{-4}
0.8	0.7773×10^{-2}	0.6168×10^{-4}	0.3972×10^{-2}	0.4682×10^{-4}
0.9	0.1116×10^{-1}	0.6926×10^{-4}	0.4624×10^{-2}	0.4982×10^{-4}
1.0	0.1542×10^{-1}	0.7694×10^{-4}	0.4957×10^{-2}	0.5461×10^{-4}

Table 6. Results from ADM, LADM, VIM, LVIM in Example 3 for error of q.

t	$\lvert Q_2(t)-q(t)\rvert$	$\lvert \bar{Q}_2(t)-q(t)\rvert$	$\lvert q_2(t)-q(t)\rvert$	$\lvert \bar{q}_2(t)-q(t)\rvert$
0.1	0.1008×10^{-3}	0.7705×10^{-5}	0.2490×10^{-3}	0.1546×10^{-3}
0.2	0.1181×10^{-2}	0.1970×10^{-4}	0.3173×10^{-3}	0.1786×10^{-3}
0.3	0.4276×10^{-2}	0.4726×10^{-4}	0.6642×10^{-3}	0.1562×10^{-3}
0.4	0.1022×10^{-1}	0.6680×10^{-4}	0.3122×10^{-2}	0.1236×10^{-3}
0.5	0.1933×10^{-1}	0.7788×10^{-4}	0.6722×10^{-2}	0.9414×10^{-4}
0.6	0.3108×10^{-1}	0.8213×10^{-4}	0.1064×10^{-1}	0.7116×10^{-4}
0.7	0.4406×10^{-1}	0.8156×10^{-4}	0.1395×10^{-1}	0.5440×10^{-4}
0.8	0.5618×10^{-1}	0.7805×10^{-4}	0.1587×10^{-1}	0.4259×10^{-4}
0.9	0.6507×10^{-1}	0.7315×10^{-4}	0.1585×10^{-1}	0.3444×10^{-4}
1.0	0.6868×10^{-1}	0.6809×10^{-4}	0.1372×10^{-1}	0.2897×10^{-4}

Table 7. Results from ADM, LADM, VIM, LVIM in Example 4 for error of p.

t	$\lvert P_2(t)-p(t)\rvert$	$\lvert \bar{P}_2(t)-p(t)\rvert$	$\lvert p_2(t)-p(t)\rvert$	$\lvert \bar{p}_2(t)-p(t)\rvert$
0.2	0.6882×10^{-6}	0.7474×10^{-7}	0.1835×10^{-5}	0.2075×10^{-6}
0.4	0.1487×10^{-4}	0.9987×10^{-6}	0.7068×10^{-4}	0.2977×10^{-5}
0.6	0.9982×10^{-4}	0.3868×10^{-5}	0.5451×10^{-3}	0.1133×10^{-4}
0.8	0.4343×10^{-3}	0.1029×10^{-4}	0.2213×10^{-2}	0.2755×10^{-4}
1.0	0.1453×10^{-2}	0.2267×10^{-4}	0.6289×10^{-2}	0.5269×10^{-4}
1.2	0.3966×10^{-2}	0.4405×10^{-4}	0.1416×10^{-1}	0.8629×10^{-4}
1.4	0.9139×10^{-2}	0.7736×10^{-4}	0.2702×10^{-1}	0.1265×10^{-3}
1.6	0.1825×10^{-1}	0.1246×10^{-3}	0.4547×10^{-1}	0.1708×10^{-3}
1.8	0.3226×10^{-1}	0.1859×10^{-3}	0.6923×10^{-1}	0.2164×10^{-3}
2.0	0.5135×10^{-1}	0.2597×10^{-3}	0.9713×10^{-1}	0.2618×10^{-3}

Table 8. Results from ADM, LADM, VIM, LVIM in Example 4 for error of q.

t	$\lvert Q_2(t)-q(t)\rvert$	$\lvert \bar{Q}_2(t)-q(t)\rvert$	$\lvert q_2(t)-q(t)\rvert$	$\lvert \bar{q}_2(t)-q(t)\rvert$
0.2	0.1978×10^{-5}	0.2943×10^{-7}	0.8801×10^{-6}	0.2836×10^{-6}
0.4	0.2878×10^{-4}	0.4891×10^{-6}	0.3443×10^{-4}	0.3670×10^{-5}
0.6	0.1359×10^{-3}	0.2345×10^{-5}	0.2607×10^{-3}	0.1288×10^{-4}
0.8	0.3962×10^{-3}	0.7342×10^{-5}	0.1035×10^{-2}	0.2859×10^{-4}
1.0	0.8658×10^{-3}	0.1772×10^{-4}	0.2865×10^{-2}	0.4921×10^{-4}
1.2	0.1550×10^{-2}	0.3536×10^{-4}	0.6257×10^{-2}	0.7136×10^{-4}
1.4	0.2418×10^{-2}	0.6076×10^{-4}	0.1152×10^{-1}	0.9092×10^{-4}
1.6	0.3502×10^{-2}	0.9244×10^{-4}	0.1861×10^{-1}	0.1043×10^{-3}
1.8	0.5055×10^{-2}	0.1270×10^{-3}	0.2702×10^{-1}	0.1097×10^{-3}
2.0	0.7731×10^{-2}	0.1604×10^{-3}	0.3586×10^{-1}	0.1076×10^{-3}

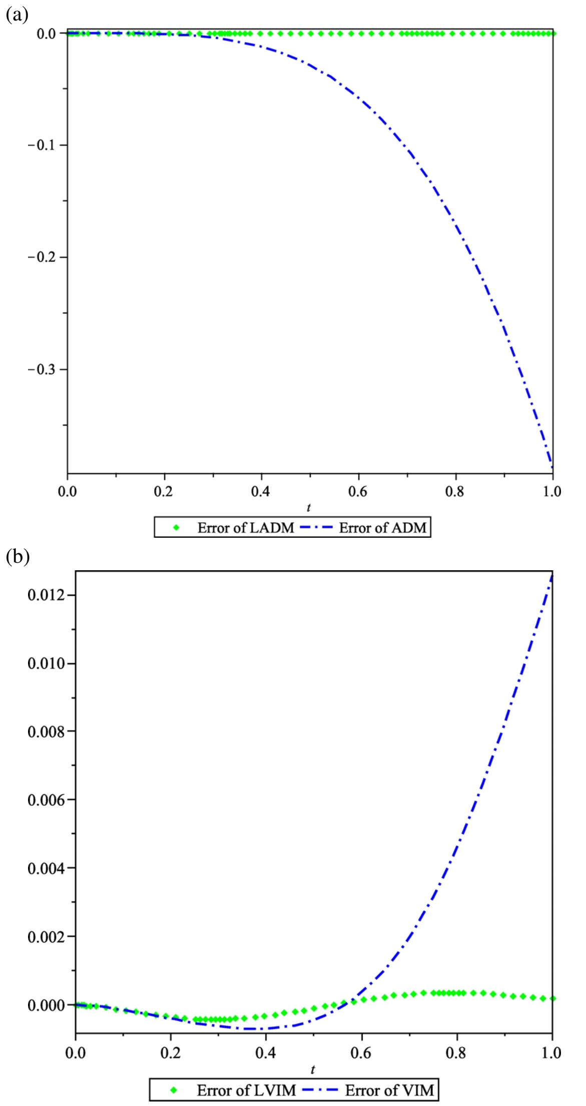

Fig. 2 (colour online). Plot of $Q_2(t)-(t^2-t)$ and $\bar{Q}_2(t)-(t^2-t)$ (a) for the Adomian method and LADM with $h=0.001$ also, $q_2(t)-(t^2-t)$ and $\bar{q}_2(t)-(t^2-t)$ (b) for VIM and the new technique (LVIM) with $h=0.001$.

Using LADM, we implement this method on this system with $h=0.001$ and $n=2$.

Using VIM, we found two iterations of the sequence. With $n=2$ we obtain p_2 and q_2 as the approximate solutions. In Figure 1, the errors of the approximation of p by ADM and LADM with $n=2$ and $h=0.001$ are presented. Also, the errors of VIM $(p_2(t)-p)$ with $n=2$ and LVIM $(\bar{p}_2(t)-p)$ with $n=2$ and $h=0.001$ have been plotted.

In Figure 2, the errors of approximation of q with the above information have been plotted. In Tables 1 and 2, the results from ADM, LADM, VIM, and LVIM are shown. In this example the polynomials g_1 and g_2 are affected to errors of p and q. The error increases rapidly as t grows, because g_1 and g_2 converge to infinity as t increases.

5.2. Example 2.

In this example we consider

$$f_1(t)=2t-3,\quad f_2(t)=t,$$
$$k_1=2,\quad k_2=2,\quad \gamma_1=1,\quad \gamma_2=1,$$
$$T_0=\frac{1}{3},\quad \alpha_1=0,\quad \alpha_2=0,$$
$$g_1(t)=$$
$$t^2\left(2-3t\mathrm{e}^{-t}-\frac{7}{2}\mathrm{e}^{-t}+\frac{13}{6}t\mathrm{e}^{\frac{1}{3}-t}+\frac{22}{9}\mathrm{e}^{\frac{1}{3}-t}\right)-2t,$$

and

$$g_2(t)=\frac{1}{648}\mathrm{e}^{-t}(342t^3-8t^2+325t+324).$$

Here, $p(t)=-t^2$ and $q(t)=\frac{1}{2}t\mathrm{e}^{-t}$ are the exact solutions.

Similar to the previous example, we consider ADM and LADM with $n=2$, and $h=0.004$. Also we get similar results for VIM and LVIM. We plotted the error of solutions p and q in Figures 3 and 4, respectively. In Tables 3 and 4, the results from ADM, LADM, VIM, and LVIM are shown.

5.3. Example 3.

In this example, we solve the system of equations (1) and (2) with

$$f_1(t)=t,\quad f_2(t)=t+1,$$
$$k_1=1,\quad k_2=1,\quad \gamma_1=\frac{1}{2},\quad \gamma_2=3,$$
$$T_0=\frac{1}{4},\quad \alpha_1=0,\quad \alpha_2=-1,$$
$$g_1(t)=2t-1-(t^2-t)\left(1+\frac{11}{18}\mathrm{e}^{-3t}-\frac{1}{36}\mathrm{e}^{\frac{3}{4}-3t}\right),$$

and

$$g_2(t)=\frac{1}{3072}\mathrm{e}^{-3t}(10080t^2-10304t+6275).$$

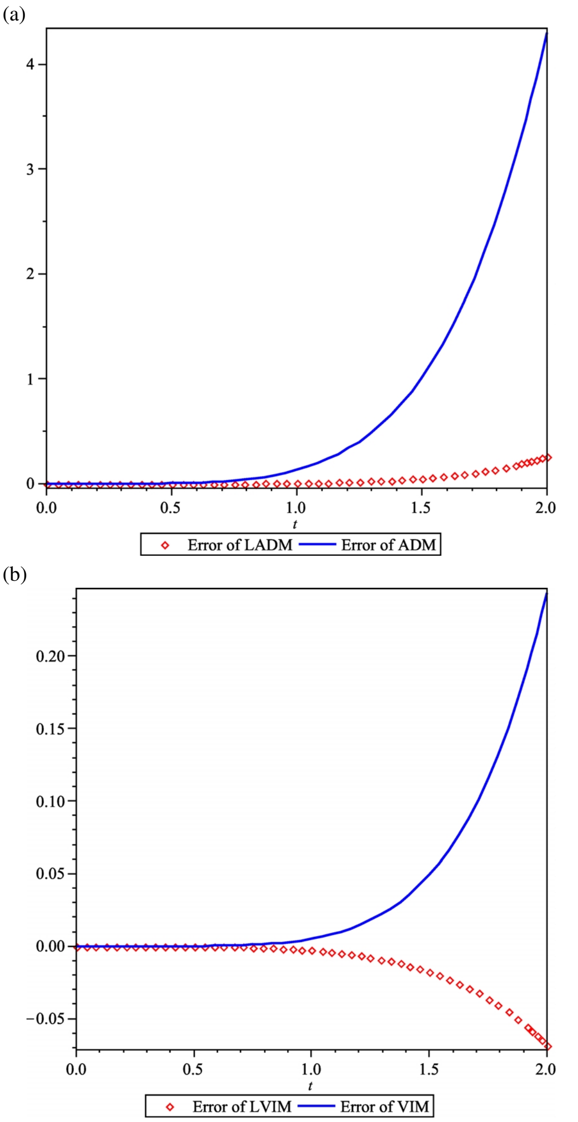

Fig. 3 (colour online). Plot of $P_2(t)-(-t^2)$ and $\bar{P}_2(t)-(-t^2)$ (a) for the Adomian method and LADM with $h=0.004$ also, $p_2(t)-(-t^2)$ and $\bar{p}_2(t)-(-t^2)$ (b) for the He's variatinal iteration method (VIM) and the new technique (LVIM) with $h=0.004$.

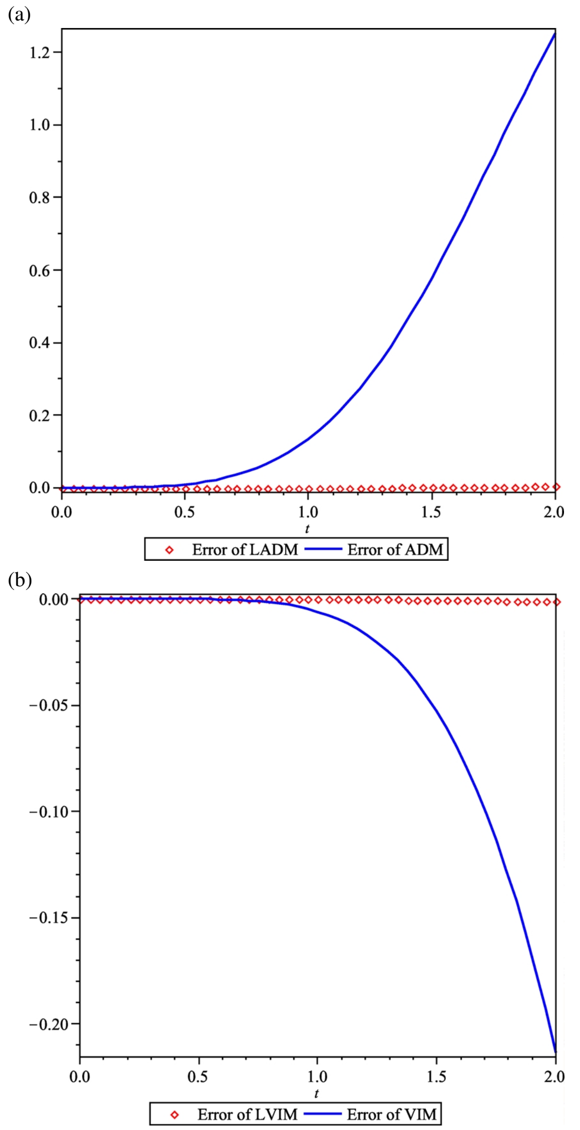

Fig. 4 (colour online). Plot of $Q_2(t)-(\frac{1}{2}t\mathrm{e}^{-t})$ and $\bar{Q}_2(t)-(\frac{1}{2}t\mathrm{e}^{-t})$ (a) for the Adomian method and LADM with $h=0.004$ also, $q_2(t)-(\frac{1}{2}t\mathrm{e}^{-t})$ and $\bar{q}_2(t)-(\frac{1}{2}t\mathrm{e}^{-t})$ (b) for VIM and the new technique (LVIM) with $h=0.004$.

Here, $p(t)=t^2-t$ and $q(t)=-\mathrm{e}^{-3t}$ are the exact solutions.

Such as previous examples, we consider ADM and LADM, with $n=2$ and $h=0.004$. Also we have similar information for VIM and LVIM. The efficiency of the new methods is clear from Figures 5 and 6. In Tables 5 and 6, the results from ADM, LADM, VIM, and LVIM are shown.

5.4. Example 4.

As the last example, we solve the system of equations (1) and (2) with

$$f_1(t)=1, \quad f_2(t)=\mathrm{e}^{-t},$$

$$k_1=\frac{1}{3}, \quad k_2=\frac{1}{2}, \quad \gamma_1=2, \quad \gamma_2=1,$$

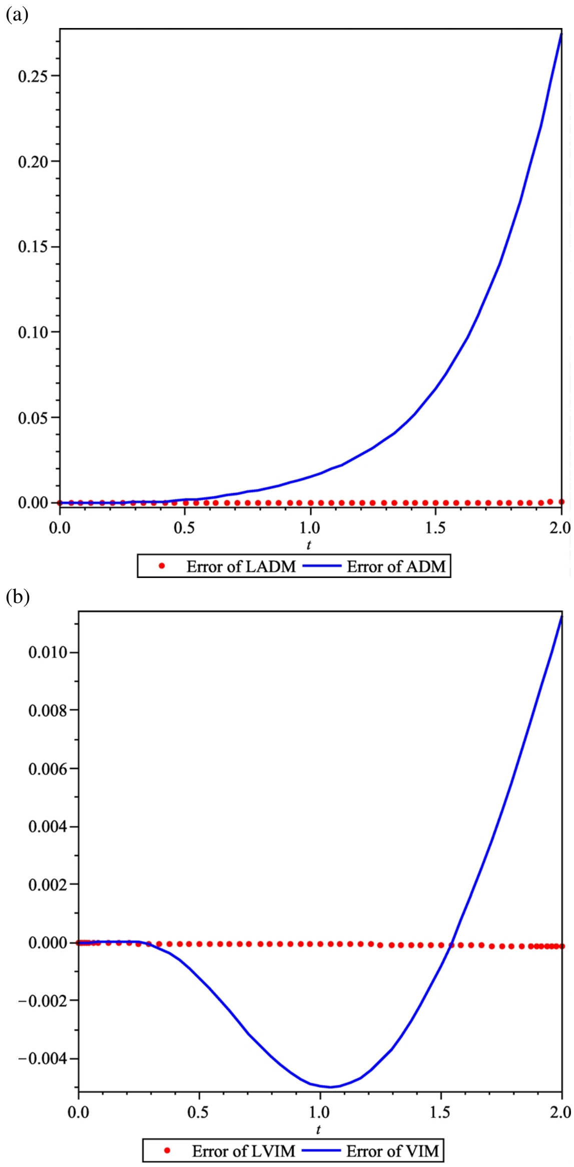

Fig. 5 (colour online). Plot of $P_2(t) - (t^2 - t)$ and $\bar{P}_2(t) - (t^2 - t)$ (a) for the Adomian method and LADM with $h = 0.004$ also, $p_2(t) - (t^2 - t)$ and $\bar{p}_2(t) - (t^2 - t)$ (b) for the He's variational iteration method (VIM) and the new technique (LVIM) with $h = 0.004$.

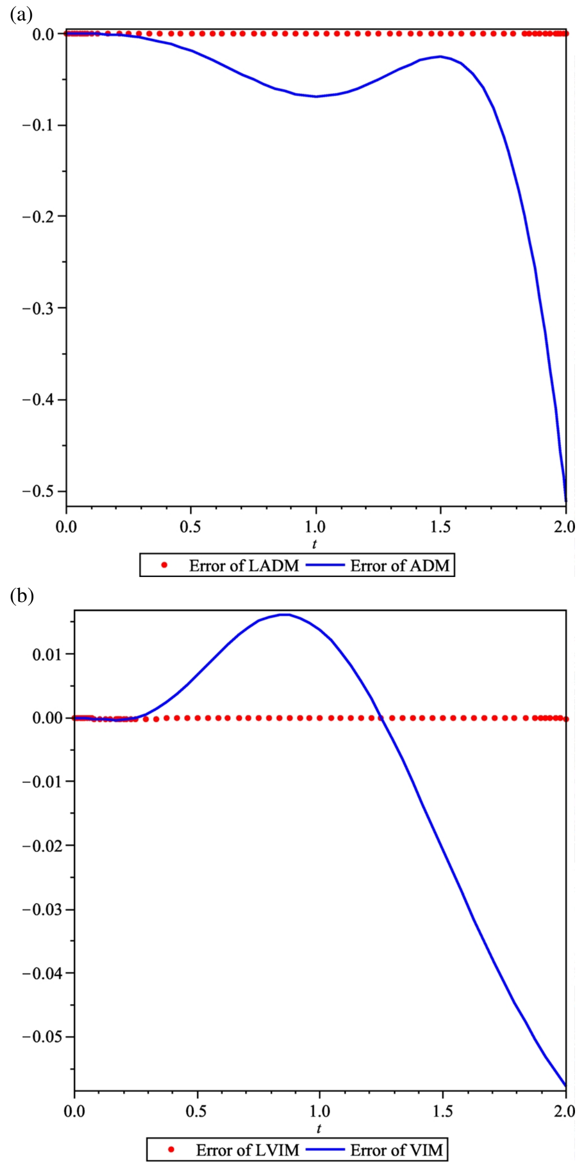

Fig. 6 (colour online). Plot of $Q_2(t) - (-\mathrm{e}^{-3t})$ and $\bar{Q}_2(t) - (-\mathrm{e}^{-3t})$ (a) for the Adomian method and LADM with $h = 0.004$ also, $q_2(t) - (-\mathrm{e}^{-3t})$ and $\bar{q}_2(t) - (-\mathrm{e}^{-3t})$ (b) for the He's variational iteration method (VIM) and the new technique (LVIM) with $h = 0.004$.

$$T_0 = \frac{3}{10}, \quad \alpha_1 = 0, \quad \alpha_2 = 0,$$

$$g_1(t) = \frac{1}{4}\cos(t) - \frac{1}{4}\sin(t)\left[\frac{1}{3} + \frac{1}{2}\sin(t) - \frac{1}{4}\cos(t) + \frac{1}{4}\cos\left(t - \frac{3}{10}\right)\right],$$

and

$$g_2(t) = -\frac{1}{4}\cos(t) + \frac{1}{4}\sin(t)\left\{-\frac{1}{2} + \frac{3}{8}\sin(t) - \frac{1}{8}\cos(t) + \frac{1}{8}\mathrm{e}^{-\frac{3}{10}}\left[\cos\left(t - \frac{3}{10}\right) - \sin\left(t - \frac{3}{10}\right)\right]\right\}.$$

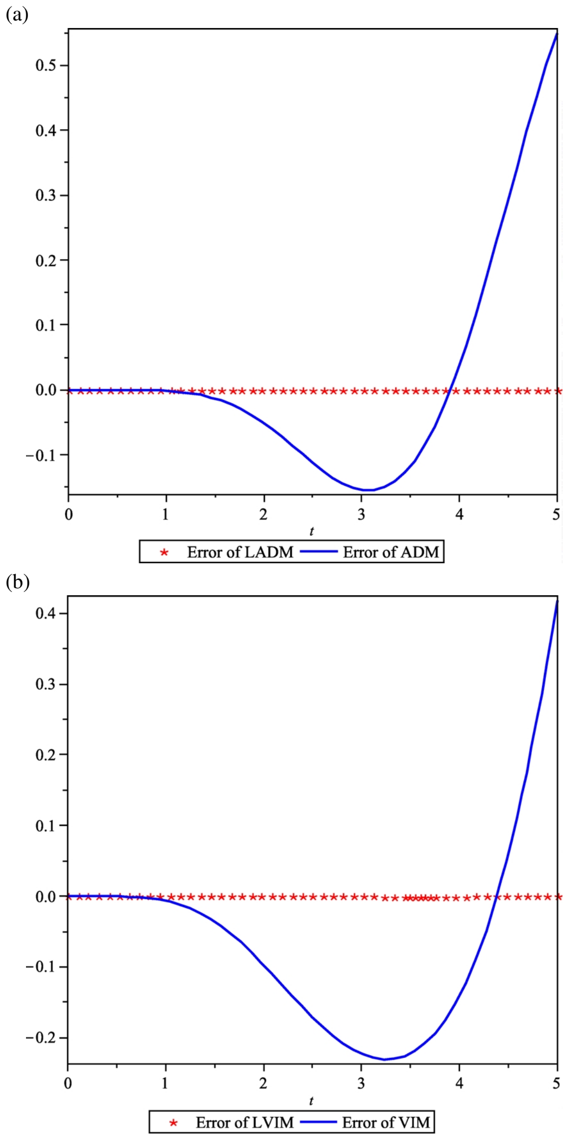

Fig. 7 (colour online). Plot of $P_2(t)-(\frac{1}{4}\sin(t))$ and $\bar{P}_2(t)-(\frac{1}{4}\sin(t))$ (a) for the Adomian method and LADM with $h=0.005$ also, $p_2(t)-(\frac{1}{4}\sin(t))$ and $\bar{p}_2(t)-(\frac{1}{4}\sin(t))$ (b) for the He's variational iteration method (VIM) and the new technique (LVIM) with $h=0.005$.

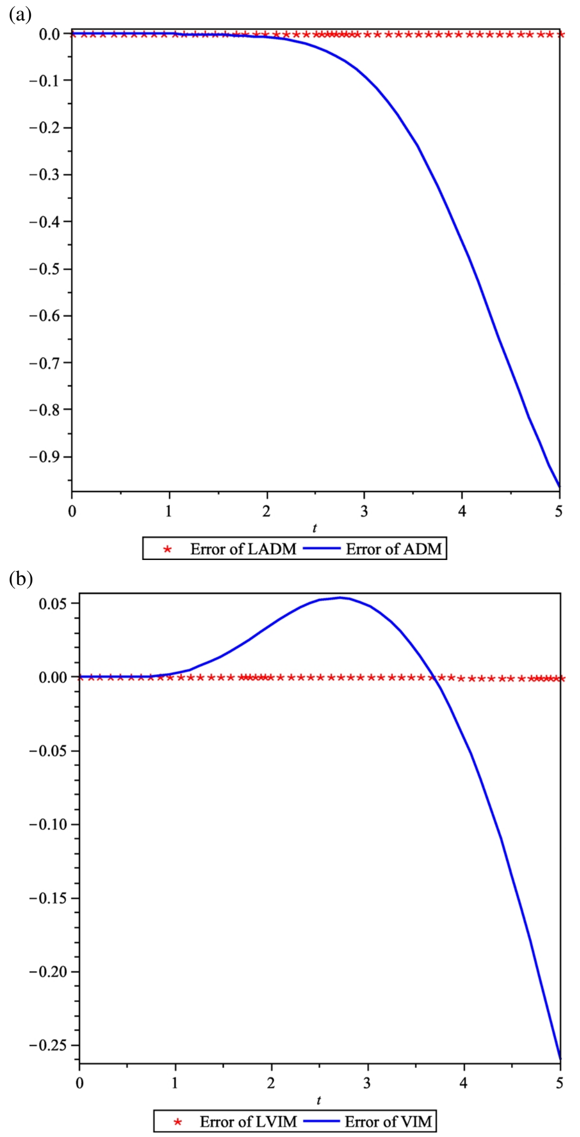

Fig. 8 (colour online). Plot of $Q_2(t)-(-\frac{1}{4}\sin(t))$ and $\bar{Q}_2(t)-(-\frac{1}{4}\sin(t))$ (a) for the Adomian method and LADM with $h=0.005$ also, $q_2(t)-(-\frac{1}{4}\sin(t))$ and $\bar{q}_2(t)-(-\frac{1}{4}\sin(t))$ (b) for the He's variational iteration method (VIM) and the new technique (LVIM) with $h=0.005$.

Here, $p(t)=\frac{1}{4}\sin(t)$ and $q(t)=-\frac{1}{4}\sin(t)$ are the exact solutions.

Using ADM and LADM with $n=2$ and $h=0.005$, also using VIM and LVIM, we plotted errors in Figures 7 and 8. Additionally, the numerical results are shown in Tables 7 and 8. It is clear from these tables that the use of the new methods provides more accurate solution in a larger domain.

6. Conclusion

The Adomian decomposition [10–15] and variational iteration techniques [21–25] are efficient methods for solving various kinds of problems. The main advantage of these methods is that they do not require discretization of the variables [36]. Also, these are not affected by computation round off errors. In this paper, these techniques are implemented in a new way, which is based on the successive use of these methods in smaller regions. The results show the efficiency of the new techniques for solving integro-differential equations. The main idea behind the approach in the current paper can be employed to solve ordinary differential or partial differential equations to improve the results of applying the classic Adomian decomposition method or the classic variational iteration method. It is worth to mention that the use of the new approach to improve the homotopy perturbation method [1, 8, 9, 37] or the homotopy analysis method [38, 39] can be a useful investigation.

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